Exact Eigenstates for Repulsive Bosons in Two Dimensions

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We consider a model of N two-dimensional bosons in a harmonic potential with weak repulsive delta-function interactions. We show analytically that, for angular momentum $L \leq N$, the elementary symmetric polynomials of particle coordinates measured from the center of mass are exact eigenstates with energy $N(N - L/2 - 1)$. Extensive numerical analysis confirms that these are actually the ground states, but we are currently unable to prove this analytically. The special case $L = N$ can be thought of as the generalisation of the usual superfluid one-vortex state to Bose-Einstein condensates in a trap.

I. INTRODUCTION

The recent observation of Bose-Einstein condensation in dilute gases of alkali metals [1–3] has stimulated much interest in the properties of systems of interacting bosons. A question of particular interest is whether such systems will form vortices under rotation, as occurs in superfluid ^{4}He. Experimentally such vortex states have been observed, both in two component systems [4] and in a stirred condensate of ^{87}Rb atoms [5]. Theoretically the stability of such vortices has been considered both in the Thomas-Fermi limit of strong interactions between atoms using the Gross-Pitaevskii equation [6,7], and also in the weak interaction limit [9–15] where the coherence length is much larger than the size of the atom cloud. It is the latter limit we shall focus upon in this paper.

Pursuing an analogy with the fractional quantum Hall effect [8], we previously introduced a model of weakly interacting bosons in a harmonic well [9] to address the question of whether attractive bosons condense. We proved analytically that all the angular momentum in this model is carried by the center of mass for attractive bosons, whereas for repulsive bosons we numerically found that a vortex state forms in the case of one unit of angular momentum per boson. Further numerical work by one of the authors [10,11] extensively investigated the properties of ground states of the repulsive model for angular momentum $L > N$, demonstrating that although such states are more complicated than the analytic ones known for $L \leq N$, they can still be understood either within vortex or composite fermion or boson pictures. Mottelson [12] considered the low-lying eigenstates for the case $L \leq N$, whilst Bertsch and Papenbrock [13] performed numerical computations and noted that the ground state of the repulsive model for $L \leq N$ is the elementary symmetric polynomial, $\tilde{e}_L$, of coordinates, $z_i = x_i + iy_i$, relative to the center of mass, $R = \sum_i z_i/N$. Finally, recent work by Kavoulakis et al [14] and Jackson et al [15] has considered the relationship between mean and exact numerical solutions in the limit of large N.

In this Letter we present an analytical proof that the state described above is an exact eigenstate of the model for $L \leq N$. We have unfortunately been unable to show that this state is the ground state, although we have considerable numerical evidence to show that this is the case.

The model is of N bosons in a harmonic potential in two dimensions interacting via a delta–function potential, for which the Hamiltonian is

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\nabla_i^2 + \frac{1}{2}m\omega^2 r_i^2 \right] + V \sum_{i<j} \delta(\mathbf{r_i} - \mathbf{r_j}) \quad (1)$$

The natural way to look at this problem is in second quantised form, so first we must solve the non-interacting problem. We do this in plane polar coordinates since we are interested in angular momentum properties. The one-particle wavefunctions and energies are then

$$\begin{aligned} \psi_{n_r,\ell} &= N_{n_r,\ell} R(n_r, |\ell|, -r^2/2) r^\ell e^{i\ell\theta} \\ E_{n_r,\ell} &= (n_r + |\ell| + 1)\hbar\omega \end{aligned} \quad (2)$$

where $R(n, l, x)$ is the confluent hypergeometric function, n_r is the radial quantum number and ℓ is the angular momentum. The total energy and angular momentum of a system of N non-interacting bosons in this harmonic well is thus

$$E_{tot} = \sum_{i=1}^{N} (n_{r,i} + |\ell_i| + 1)\hbar\omega, \qquad L_{tot} = \sum_{i=1}^{N} \ell_i. \quad (3)$$

The ground state manifold is then obtained by putting all bosons into the lowest radial state, $n_{r,i} = 0$, and choosing angular momentum ℓ_i all of the same sign (which we choose to be positive), such that

$$\sum_{i=1}^{N} \ell_i = L \quad \Rightarrow \quad E_{tot} = (L + N)\hbar\omega. \quad (4)$$

It can be seen that the ground state manifold has a degeneracy which is equal to the number of *partitions* of the total angular momentum L into the N angular momenta ℓ_i of the individual bosons.

If we assume that the dimensionless interaction strength is very small, $|\eta| \ll 1$, where $\eta = V/\hbar\omega$, we can treat it as a perturbation whose sole effect will be to lift the large degeneracy of the non-interacting ground state. This means we can work within the non-interacting

ground state manifold, which freezes out the one-particle part of the Hamiltonian, and consider the effect of the interaction using degenerate perturbation theory. This is essentially an extension of an approach used for fractional quantum Hall effect systems [8] to bosons. The normalised one-particle wavefunctions are given by

$$\psi_\ell(r,\theta) = \frac{1}{\sqrt{\ell!2\pi}} r^\ell e^{i\ell\theta} e^{-r^2/2} = \frac{1}{\sqrt{\ell!2\pi}} z^\ell e^{-|z|^2/2}, \quad (5)$$

where we have moved to complex notation. The second quantized form of the interaction Hamiltonian is thus

$$\hat{H} = \sum_{m_1,m_2,n_1,n_2} V_{m_1 m_2 n_1 n_2} a^+_{m_1} a^+_{m_2} a_{n_1} a_{n_2} \quad (6)$$

where the matrix element

$$V_{m_1 m_2 n_1 n_2} = \frac{\eta}{4\pi} \frac{(m_1+n_1)!}{2^{m_1+m_2}} \delta_{m_1+m_2, n_1+n_2}. \quad (7)$$

For future convenience we will set $\eta = 4\pi$.

To perform the degenerate perturbation theory for given N and L, we first need the basis states which are labelled by the partitions of L into N pieces. Let us write a partition λ in the form

$$\lambda = 0^{\lambda_0} 1^{\lambda_1} 2^{\lambda_2} \ldots = \prod_i i^{\lambda_i} \quad (8)$$

where $\sum \lambda_i = N$ and $\sum i\lambda_i = L$. The corresponding basis state is then

$$|\,\lambda\,\rangle = \left[\prod_i \frac{(a^+_i)^{\lambda_i}}{\sqrt{\lambda_i!}} \right], |\,0\,\rangle \quad (9)$$

where a^+_ℓ creates a boson of angular momentum ℓ. In coordinate space, this basis state takes the form

$$\left[\frac{\prod_i \lambda_i!}{(2\pi)^N N! \prod_i (i!)^{\lambda_i}} \right]^{1/2} m_\lambda(z_1, z_2 \ldots z_N), \quad (10)$$

where m_λ is the *monomial symmetric polynomial* corresponding to the partition λ. The latter is the symmetric polynomial of the N variables $(z_1, z_2 \ldots z_N)$ which has λ_i i-th powers.

To consider the problem either analytically or numerically requires us to calculate the elements of the symmetric interaction matrix, $H_{\lambda\mu} = \langle\lambda|\hat{V}|\mu\rangle$. This is obviously best performed using the second quantized approach. There are two types of matrix elements, diagonal and off-diagonal, and their evaluation is different. For the diagonal elements $H_{\lambda\lambda}$, we must sum over every possible pair of elements in the partition, whether distinct or identical,

$$H_{\lambda\lambda} = \sum_i \lambda_i(\lambda_i - 1) V_{\lambda_i\lambda_i\lambda_i\lambda_i} + 4 \sum_{i<j} \lambda_i \lambda_j V_{\lambda_i\lambda_j\lambda_i\lambda_j}. \quad (11)$$

The off-diagonal elements $H_{\lambda\mu}$ can only be non-zero if λ and μ differ only in the angular momenta of two particles. This can affect either 4 separate angular momentum states, as in the case where the angular momentum transfer is $0+4 \to 1+3$; or 3 separate angular momentum states, as in the case of angular momentum transfer $0+4 \to 2+2$. In terms of partitions, in case (i) $\lambda_i = \mu_i$ except at four values of i, which we call $i_1 \ldots i_4$, and $\lambda_{i_1} = \mu_{i_1} + 1$, $\lambda_{i_2} = \mu_{i_2} + 1$, $\lambda_{i_3} = \mu_{i_3} - 1$, $\lambda_{i_4} = \mu_{i_4} - 1$. The matrix element is then

$$H_{\lambda\mu} = H_{\mu\lambda} = 4\sqrt{\lambda_{i_1}\lambda_{i_2}\mu_{i_3}\mu_{i_4}} V_{i_1 i_2 i_3 i_4}. \quad (12)$$

In case (ii) $\lambda_i = \mu_i$ except at three values of i which we call i_1, i_2, i_3, and $\lambda_{i_1} = \mu_{i_1} + 2$, $\lambda_{i_2} = \mu_{i_2} - 1$, $\lambda_{i_3} = \mu_{i_3} - 1$. The matrix element is then

$$H_{\lambda\mu} = H_{\mu\lambda} = 2\sqrt{\lambda_{i_1}(\lambda_{i_1}-1)\mu_{i_2}\mu_{i_3}} V_{i_1 i_1 i_2 i_3}. \quad (13)$$

These formulas allow one to write down the interaction matrix $H_{\lambda\mu}$ which can then be diagonalised to give the energy eigenvalues and eigenstates.

II. THE SUBSPACE PROPERTY

If we look at the original Hamiltonian, we find that the interaction term depends only upon relative coordinates. To see this, change variables to the center of mass variable, $R = \sum_i z_i/N$, and $N-1$ relative coordinates such as $\tilde{z}_i = z_i - R$, where $i = 1 \ldots N-1$: the interaction is then a function only of the relative coordinates $\tilde{z}_i$. It follows that if $\psi(z_1 \ldots z_N)$ is an eigenfunction of H with a certain energy E and angular momentum L, then $R\psi(z_1 \ldots z_N)$ is an eigenfunction of H with energy E and angular momentum $L+1$. This means that all eigenfunctions of H at a given L are reproduced at all higher L, and thus the number of new states at any L is just $n_L - n_{L-1}$, the difference between the number of partitions of L and $L-1$ respectively [8].

This subspace property makes it natural to think in terms of a second type of symmetric polynomial, the *elementary symmetric polynomials*. These are defined by, for N variables,

$$e_L = \sum_{i_1 < i_2 < \ldots < i_L} z_{i_1} z_{i_2} \ldots z_{i_L} \quad (14)$$

where $L \le N$. For a general partition λ we define

$$e_\lambda = \prod_i e_i^{\lambda_i}. \quad (15)$$

The set of new eigenstates at any total angular momentum L is seen to be spanned by the $\tilde{e}_\lambda$, the elementary symmetric polynomials of the relative coordinates $\tilde{z}_i$, where we now include $\tilde{z}_N = -\tilde{z}_1 - \tilde{z}_2 \ldots \tilde{z}_{N-1}$. Since $\tilde{e}_1 = 0$, only partitions with $\lambda_1 = 0$ can be formed (i.e. partitions with no part equal to 1), which gives exactly the correct number of states.

III. PROOF THAT THE $\tilde{\mathbf{E}}_\mathbf{L}$ ARE EXACT EIGENSTATES

In this section we prove that the states $\tilde{e}_L$ for $L \leq N$ ($L \neq 1$) are eigenstates of $\hat{H}$ with eigenvalue $N(N-1-L/2)$. The proof is essentially a brute force method: we operate the Hamiltonian on the state $\tilde{e}_L$, and show that the result is the above eigenvalue times $\tilde{e}_L$. The derivation naturally falls into five steps, which we detail below.

(1) Writing $\tilde{\mathbf{e}}_\mathbf{L}$ in Terms of $\mathbf{e}_\mathbf{L}$ and R

Consider $\tilde{e}_M$, which can be written as

$$\tilde{e}_M = \sum_{i_1<i_2<\ldots<i_M} (z_{i_1} - R)\ldots(z_{i_M} - R). \tag{16}$$

If we expand out this product we will get the elementary symmetric polynomials of the z_i, namely the e_L, $L \leq M$, multiplied by R^{M-L}. To get the correct coefficients in this expansion we note that $\tilde{e}_M$ has $N!/M!(N-M)!$ terms whilst e_L has $N!/L!(N-L)!$ terms. In the expansion of $\tilde{e}_M$, each product term will produce $M!/L!(M-L)!$ terms of the type which will add up to produce $R^{M-L}e_L$, so that the coefficient of e_L in the expansion is $(N-L)!/(N-M)!(M-L)!$ It follows that

$$\tilde{e}_M = \sum_{L=2}^{M} (-1)^{M-L} \frac{N-L!}{N-M!\,M-L!} e_L R^{M-L} + (-1)^{M-1} \frac{N!(M-1)}{N-M!\,M!} R^M, \tag{17}$$

where we have noticed that, since $e_1 = R$, the last two terms have the same form and should be combined.

(2) Operating Hamiltonian $\hat{\mathbf{H}}$ on $\mathbf{e}_\mathbf{L}$

An important feature of e_L is that it is also the monomial function m_λ corresponding to $\lambda = 0^{N-L}1^L$. The normalised version of e_L can thus be written as $|\,e_L\rangle \equiv |\,0^{N-L}1^L\rangle$, and it is clear that the Hamiltonian $\hat{H}$ can only connect this state to itself and $|\,A\,\rangle \equiv |\,0^{N-L+1}1^{L-2}2^1\rangle$, where we have labelled this state as $|\,A\,\rangle$ for convenience in what follows. The two matrix elements can be calculated using from the formulas derived in the previous sections for diagonal and off-diagonal elements respectively. The diagonal element is given by

$$\langle e_L|\hat{H}|e_L\rangle == N^2 - N - \frac{1}{2}L(L-1). \tag{18}$$

The off-diagonal element is found by using the rule for the case where only three separate angular momentum states change (here $1+1 \to 0+2$) to give

$$\langle\, A\,|\,\hat{H}\,|\, e_L\rangle = \frac{1}{2}\sqrt{2L(L-1)(N-L+1)} \tag{19}$$

The final result for the operation of the Hamiltonian on e_L is thus

$$\hat{H}|\,e_L\rangle = \left[N^2 - N - \frac{1}{2}L(L-1)\right]|\,e_L\rangle + \frac{1}{2}\sqrt{2L(L-1)(N-L+1)}|\,A\rangle. \tag{20}$$

(3) Removing the Normalisation Factors

We want to get rid of the normalisation factors for the eigenstates so that the Hamiltonian will act directly on symmetric polynomials such as e_L. In the previous equation we should divide by the normalization factor for $|\,0^{N-L}1^L\rangle$ and multiply by that for $|\,0^{N-L+1}1^{L-2}2^1\rangle$. Using the form for the normalisation factors given in Eq. (10), we find that the normalised version of Eq. (20) is

$$\hat{H}|e_L) = [N^2 - N - \frac{1}{2}L(L-1)]|e_L) + \frac{N-L+1}{2}|A), \tag{21}$$

where the $|\lambda)$ are the symmetric polynomials with no normalisation factor.

(4) Relating $|\,\mathbf{0^{N-L+1}1^{L-2}2^1}\rangle$ to $\mathbf{e_L}$ and $\mathbf{Re_{L-1}}$

Consider the product

$$NR|e_{L-1}) = \left[\sum_{i=1}^{N} z_i\right] \sum_{i_1<\ldots<i_{L-1}} z_{i_1} z_{i_2} \ldots z_{i_{L-1}} \tag{22}$$

The above product can clearly only produce $|0^{N-L}1^L)$ or $|0^{N-L+1}1^{L-2}2^1)$, depending upon whether the z_j from the prefactor is not or is included in the set $(z_{i_1}, z_{i_2}\ldots z_{i_{L-1}})$. Moreover we see that each element of $|0^{N-L+1}1^{L-2}2^1)$ can only be made in one way, so that

$$NRe_{L-1} = |0^{N-L+1}1^{L-2}2^1) + Ce_L. \tag{23}$$

To find the coefficient C we just count terms: NRe_{L-1} has $(N+1)!/(N-L+1)!\,(N-1)!$ terms, $|A)$ has $N!/(N-L+1)!(L-2)!$ terms, and e_L has $N!/(N-L)!L!$ terms. This leads to the result $C = L$, and hence

$$|A) = NRe_{L-1} - Le_L. \tag{24}$$

Combining this with Eq. (21) gives

$$\hat{H}e_L = \left[N^2 - \left(1+\frac{L}{2}\right)N\right]e_L + \frac{N(N-L+1)}{2}Re_{L-1}. \tag{25}$$

(5) Operating $\hat{\mathbf{H}}$ onto $\tilde{\mathbf{e}}_\mathbf{M}$

If we now operate H onto $\tilde{e}_M$ using Eq. (17) to write $\tilde{e}_M$ in terms of e_L and R, and using Eq. (25) to operate $\hat{H}$ onto e_L, we get after a lot of tedious algebra,

$$\hat{H}\tilde{e}_M = \left[N^2 - \left(1+\frac{M}{2}\right)N\right]\tilde{e}_M. \qquad (26)$$

Note that we have to compare the coefficients of three types of term separately: (i) e_M, (ii) $R^{M-L}e_L$ for $2 \leq L < M$, and (iii) R^M. The proof can be simplified a little if we use the subspace property. Introduce a projection operator, $\hat{P}$, which removes any term containing a factor of R. Operating $\hat{P}$ onto Eq. (17) gives $\hat{P}e_L = \tilde{e}_L$, whilst operating $\hat{P}$ onto Eq. (25) gives Eq. (26).

IV. DISCUSSION AND CONCLUSIONS

We have considered N bosons in a 2D harmonic potential interacting via repulsive delta-function potentials and with fixed total angular momentum $L \leq N$. Within the "lowest Landau level" approximation, we have analytically shown that the elementary symmetric polynomial of coordinates relative to the center of mass, $\tilde{e}_L$ is an exact eigenstate of this Hamiltonian with eigenvalue $N(N-L/2-1)$. Extensive numerical analysis shows that this state is actually the ground state. This is not surprising since in the special case $L = N$ this is just the one-vortex state discussed in Ref. [9] which is expected to be the ground state by analogy to superfluid ^{4}He. One can also see that there is a sense in which $\tilde{e}_L$ distributes the angular momentum equally between particles subject to the subspace property. We have attempted to prove analytically that $\tilde{e}_L$ is the ground state, but have so far failed. The situation is much harder than in the proof we presented in Ref. [9] to show that the eigenstate with the largest eigenvalue corresponds to all angular momentum being in the center of mass motion. The main problem is that the eigenvector for the smallest eigenvalue has components of both signs to reduce its energy, and frustration results between any set of three basis states that are connected by $\hat{H}$: the difficulty is essentially that of a quantum antiferromagnet compared to a ferromagnet. For completeness we hope that the state considered in this paper will be analytically shown to be the ground state in the future, but from numerics there can be little doubt that it is.

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[1] M.H. Anderson, J. Ensher, M. Matthews, C. Wieman and E. Cornell, Science **269**, 198 (1995).

[2] K.B. Davis, M.O. Mewes, N. van Druten, D. Durfee, D. Kurn and W. Ketterle, Phys. Rev. Lett. bf 75, 3969 (1995).

[3] C.C. Bradley, C.A. Sackett and R.G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).

[4] M.R. Matthews, B.P. Anderson, P.C. Haljan, D.S. Hall, C.E. Wieman and E.A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).

[5] K. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000)¿

[6] D.S. Rokhsar, Phys. Rev. Lett. **79**, 2164 (1997).

[7] D.A. Butts and D.S. Rokhsar, Nature **397**, 327 (1999).

[8] S.A. Trugman and S. Kivelson, Phys. Rev. **B** 28, 4506 (1985).

[9] N.K. Wilkin, J.M.F. Gunn and R.A. Smith, Phys. Rev. Lett. **80**, 2265 (1998).

[10] N.R. Cooper and N.K. Wilkin, Phys. Rev. B **60**, R16279 (1999).

[11] N.K. Wilkin and J.M.F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).

[12] B. Mottelson, Phys. Rev. Lett. **83**, 2695 (1999).

[13] G.F. Bertsch and T. Papenbrock, Phys. Rev. Lett. **83**, 5412 (1999).

[14] G.M. Kavoulakis, B. Mottelson and C.J. Pethick, eprint cond-mat/0004307.

[15] A.D. Jackson, G.M. Kavoulakis, B. Mottelson and S.M. Reimann, eprint cond-mat/0004309.